FOLD COBORDISMS AND STABLE HOMOTOPY GROUPS

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Abstract. We prove that for $n \geq 1$ and $q > 0$ the (oriented) cobordism group of fold maps of (oriented) $(n+q)$-dimensional manifolds into $\mathbb{R}^n$ contains the direct sum of $\lfloor (q+1)/2 \rfloor$ copies of the $(n-1)$th stable homotopy group of spheres as a direct summand. We also prove that for $k \geq 1$ and $q = 2k-1$ the cobordism group of fold maps of unoriented $(n+q)$-dimensional manifolds into $\mathbb{R}^n$ also contains the nth stable homotopy group of the space $\mathbb{R}P^\infty$ as a direct summand. We have the analogous results about bordism groups of fold maps as well.

1. Introduction

In the study of singular maps one basic approach is classifying singular maps under the equivalence relation *cobordism* and describing *cobordism groups* of singular maps [2, 4, 9, 10, 14, 15, 16, 17, 18, 19].

This paper is about fold maps of $(n+q)$-dimensional manifolds into oriented n-dimensional manifolds. Fold maps of $(n+q)$-dimensional manifolds into n-dimensional manifolds have the formula $f(x_1, \ldots, x_{n+q}) = (x_1, \ldots, x_{n-1}, \pm x_n^2 \pm \cdots \pm x_{n+q}^2)$ as a local form around each singular point, and the subset of the singular points in the source manifold is a $(q+1)$-codimensional submanifold. Moreover, if we restrict a fold map to the set of its singular points, then we obtain a codimension one immersion into the target manifold of the fold map. These properties give the possibility to define simple geometrical invariants of cobordisms of fold maps (see Definition 2.1) via immersions of the singular sets. And by [20] these geometrical invariants show a strong relation with stable homotopy groups of Thom spaces of vector bundles like the circle S^1 and the projective space $\mathbb{R}P^\infty$. Furthermore, it turns out that these geometrical invariants detect an important torsion part of the cobordism group of fold maps, namely the stable homotopy groups of spheres and the space $\mathbb{R}P^\infty$.

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In [2] Ando showed that the cobordism group of fold maps of oriented n-dimensional manifolds into the n-dimensional Euclidean space $\mathbb{R}^n$ is isomorphic to the nth stable homotopy group of spheres. In this paper we are looking for similar results in the case of the cobordism group of fold maps of $(n+q)$-dimensional manifolds into $\mathbb{R}^n$, where $q > 0$. The case $n = 1$ (i.e., the cobordism group of Morse functions) has been computed by Ikegami [4].

By recent results of Ando [1], Sadykov [12] and Szűcs [19], the cobordism groups of singular maps are equal to homotopy groups of spectra. Moreover, the rank, the torsion part for sufficiently high primes (and estimations for arbitrary primes) of the homotopy groups of these spectra can be computed in special and important cases in positive codimension [19] and very probably in negative codimension [8, 13]. Our results in this paper show that computing the p-torsion of the cobordism groups of singular maps could be extremely difficult and hopeless for a small prime p. We note that in special dimension pairs [3] computes the cobordism groups of fold maps completely.

The paper is organized as follows. In Section 2 we give several basic definitions and notations. In Section 2.4 we state our main results. In Section 3 we prove our main theorems. In Section 4 we give analogous results about bordism groups of fold maps.

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1.1. **Notations.** In this paper the symbol "$\amalg$" denotes the disjoint union, for any number x the symbol "$\lfloor x \rfloor$" denotes the greatest integer i such that $i \leq x$, γ^1 denotes the universal line bundle over $\mathbb{R}P^\infty$ and the symbol ε^1 denotes the trivial line bundle over a point. The symbol $\mathrm{Imm}^{\xi^k}(n-k,k)$ denotes the cobordism group of k-codimensional immersions into $\mathbb{R}^n$ whose normal bundles can be induced from the vector bundle ξ^k (this group is isomorphic to $\pi_n^s(T\xi^k)$ where $T\xi^k$ denotes the Thom space of the bundle ξ^k [20]). The symbol $\mathrm{Imm}(n-k,k)$ denotes the cobordism group $\mathrm{Imm}^{\gamma^k}(n-k,k)$ where γ^k is the universal bundle for k-dimensional real vector bundles. The symbol $\pi_n^s(X)$ (π_n^s) denotes the nth stable homotopy group of the space X (resp. spheres). The symbol "id_A" denotes the identity map of the space A. The symbol ε refers to a small positive number. All manifolds and maps are smooth of class C^∞.

2. Preliminaries

2.1. **Fold maps.** Let $n \geq 1$ and $q > 0$. Let Q^{n+q} and N^n be smooth manifolds of dimensions $n+q$ and n respectively. Let $p \in Q^{n+q}$ be a singular point of a smooth map $f\colon Q^{n+q} \to N^n$. The smooth map f has a *fold singularity of index* λ at the singular

point p if we can write f in some local coordinates around p and $f(p)$ in the form

$$f(x_1,\ldots,x_{n+q}) = (x_1,\ldots,x_{n-1},-x_n^2-\cdots-x_{n+\lambda-1}^2+x_{n+\lambda}^2+\cdots x_{n+q}^2)$$

for some λ $(0\leq\lambda\leq q+1)$ (the index λ is well-defined if we consider that λ and $q+1-\lambda$ represent the same index).

A smooth map $f\colon Q^{n+q}\to N^n$ is called a *fold map* if f has only fold singularities.

A smooth map $f\colon Q^{n+q}\to N^n$ has a *definite fold singularity* at a fold singularity $p\in Q^{n+q}$ if $\lambda=0$ or $\lambda=q+1$, otherwise f has an *indefinite fold singularity of index* λ at the fold singularity $p\in Q^{n+q}$.

Let $S_\lambda(f)$ denote the set of fold singularities of index λ of f in Q^{n+q}. Note that $S_\lambda(f)=S_{q+1-\lambda}(f)$. Let S_f denote the set $\bigcup_\lambda S_\lambda(f)$.

Note that the set S_f is an $(n-1)$-dimensional submanifold of the manifold Q^{n+q}.

Note that each connected component of the manifold S_f has its own index λ if we consider that λ and $q+1-\lambda$ represent the same index.

Note that for a fold map $f\colon Q^{n+q}\to\mathbb{R}^n$ and for an index λ $(0\leq\lambda\leq\lfloor(q-1)/2\rfloor$ or $q+1-\lfloor(q-1)/2\rfloor\leq\lambda\leq q+1)$ the codimension one immersion $f\,|_{S_\lambda(f)}\colon S_\lambda(f)\to\mathbb{R}^n$ of the singular set of index λ $S_\lambda(f)$ has a canonical framing (i.e., trivialization of the normal bundle) by identifying canonically the set of fold singularities of index λ $(0\leq\lambda\leq\lfloor(q-1)/2\rfloor$ or $q+1-\lfloor(q-1)/2\rfloor\leq\lambda\leq q+1)$ of the map f with the fold germ $(x_1,\ldots,x_{n+q})\mapsto(x_1,\ldots,x_{n-1},-x_n^2-\cdots-x_{n+\lambda-1}^2+x_{n+\lambda}^2+\cdots x_{n+q}^2)$ $(0\leq\lambda\leq\lfloor(q-1)/2\rfloor)$ (if we consider that λ and $q+1-\lambda$ represent the same index), see, for example, [11].

If $f\colon Q^{n+q}\to N^n$ is a fold map in general position, then the map f restricted to the singular set S_f is a general positional codimension one immersion into the target manifold N^n.

Since every fold map is in general position after a small perturbation, and we study maps under the equivalence relations *cobordism* and *bordism* (see Definitions 2.1 and 4.1 respectively), in this paper we can restrict ourselves to studying fold maps which are in general position. Without mentioning we suppose that a fold map f is in general position.

2.2. **Stein factorization.** We use the notion of the Stein factorization of a smooth map $f\colon Q^q\to N^n$, where Q^q and N^n are smooth manifolds of dimensions q and n respectively $(q\geq n)$. Two points $p_1,p_2\in Q^q$ are *equivalent* if p_1 and p_2 lie on the same component of an f-fiber. Let W_f denote the quotient space of Q^q with respect to this equivalence relation and $q_f\colon Q^q\to W_f$ the quotient map. Then there exists a unique continuous map $\bar f\colon W_f\to N^n$ such that $f=\bar f\circ q_f$. The space W_f or the factorization of the map f into the composition of q_f and $\bar f$ is called the *Stein factorization* of the map f. We call the map $\bar f$ the *Stein factorization* of the map f as well. Note that if f is a generic smooth

map of a closed q-dimensional manifold into N^n (for example, if f is a fold map in general position), then its Stein factorization W_f is a compact n-dimensional CW complex.

2.3. **Cobordisms of fold maps.**

Definition 2.1. (Cobordism) Two fold maps $f_i\colon Q_i^{n+q} \to N^n$ $(i = 0, 1)$ from closed (oriented) $(n+q)$-dimensional manifolds Q_i^{n+q} $(i = 0, 1)$ into an n-dimensional manifold N^n are *(oriented) cobordant* if

a) there exists a fold map $F\colon X^{n+q+1} \to N^n \times [0,1]$ from a compact (oriented) $(n+q+1)$-dimensional manifold X^{n+q+1},
b) $\partial X^{n+q+1} = Q_0^{n+q} \amalg (-)Q_1^{n+q}$ and
c) $F\,|_{Q_0^{n+q} \times [0,\varepsilon)} = f_0 \times \mathrm{id}_{[0,\varepsilon)}$ and $F\,|_{Q_1^{n+q} \times (1-\varepsilon,1]} = f_1 \times \mathrm{id}_{(1-\varepsilon,1]}$, where $Q_0^{n+q} \times [0,\varepsilon)$ and $Q_1^{n+q} \times (1-\varepsilon, 1]$ are small collar neighbourhoods of ∂X^{n+q+1} with the identifications $Q_0^{n+q} = Q_0^{n+q} \times \{0\}$ and $Q_1^{n+q} = Q_1^{n+q} \times \{1\}$.

We call the map F a *cobordism* between f_0 and f_1.

This clearly defines an equivalence relation on the set of fold maps from closed (oriented) $(n+q)$-dimensional manifolds into an n-dimensional manifold N^n.

Let us denote the (oriented) cobordism classes of fold maps into the Euclidean space $\mathbb{R}^n$ under (oriented) fold cobordisms by $\mathcal{C}ob_f^{(O)}(n+q, -q)$. We can define a commutative group operation in the usual way on this set of (oriented) cobordism classes by the far away disjoint union.

Remark 2.2. Ikegami and Saeki [5] showed that the group $\mathcal{C}ob_f^{O}(2,-1)$ is isomorphic to $\mathbb{Z}$. The author proved that $\mathcal{C}ob_f(2,-1) = \mathbb{Z} \oplus \mathbb{Z}_2$ [6] and the group $\mathcal{C}ob_f^{O}(3,-1)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ [7]. Ikegami computed the cobordism groups $\mathcal{C}ob_f^{(O)}(p, 1-p)$ of Morse functions of p-dimensional manifolds for every $p > 0$ [4].

For $k \geq 0$ there are many results concerning the cobordism groups $\mathcal{C}ob_\tau(q,k)$, where τ is a set of singularity types, the elements of the group $\mathcal{C}ob_\tau(q,k)$ are cobordism classes of smooth maps of q-dimensional manifolds into $\mathbb{R}^{q+k}$ with only singularities in τ, and a cobordism between two such maps has only singularities in τ. See, for example, [2, 3, 9, 10, 14, 15, 16, 17, 18, 19].

2.4. **Main results.** Now we are ready to state our main theorems.

Recall that the oriented cobordism group of immersions of $(n-1)$-dimensional manifolds into $\mathbb{R}^n$ denoted in this paper by $\mathrm{Imm}^{\varepsilon^1}(n-1,1)$ is isomorphic to the stable homotopy group π_{n-1}^s [20]. In the following theorems we identify the group $\mathrm{Imm}^{\varepsilon^1}(n-1,1)$ with the group π_{n-1}^s.

Theorem 2.3. *For $n \geq 1$ and $q > 0$ the cobordism group $\mathcal{C}ob_f^{(O)}(n+q,-q)$ of fold maps of (oriented) $(n+q)$-dimensional manifolds into $\mathbb{R}^n$ contains the direct sum $\bigoplus_0^{\lfloor (q-1)/2 \rfloor} \pi_{n-1}^s$ as a direct summand.*

This direct sum $\bigoplus_0^{\lfloor (q-1)/2 \rfloor} \pi_{n-1}^s$ is detected by the homomorphisms $\varphi_0, \ldots, \varphi_{\lfloor (q-1)/2 \rfloor}$, where $\varphi_j \colon \mathcal{C}ob_f^{(O)}(n+q,-q) \to \mathrm{Imm}^{\varepsilon^1}(n-1,1)$ maps a fold cobordism class $[f]$ to the cobordism class of the framed immersion of the singular set of index j of the fold map f $(j = 0, \ldots, \lfloor (q-1)/2 \rfloor)$.

Theorem 2.4. *For $n \geq 1$, $q > 0$, $k \geq 1$ and $q = 2k-1$ the cobordism group $\mathcal{C}ob_f(n+q,-q)$ of fold maps of unoriented $(n+q)$-dimensional manifolds into $\mathbb{R}^n$ contains the direct sum $\pi_n^s(\mathbb{R}P^\infty) \oplus \bigoplus_0^{\lfloor (q-1)/2 \rfloor} \pi_{n-1}^s$ as a direct summand.*

The direct summand $\pi_n^s(\mathbb{R}P^\infty)$ is detected by the homomorphism $\psi \colon \mathcal{C}ob_f(n+q,-q) \to \mathrm{Imm}(n-1,1)$, where ψ maps a fold cobordism class $[f]$ to the cobordism class of the immersion of the singular set of index k of the fold map f.

Analogous results about bordism groups of fold maps can be found in Section 4.

3. Proof of main theorems

Proof of Theorem 2.3. For $0 \leq j \leq \lfloor (q-1)/2 \rfloor$ let us define homomorphisms $\alpha_j \colon \mathrm{Imm}^{\varepsilon^1}(n-1,1) \to \mathcal{C}ob_f^{(O)}(n+q,-q)$ and $\varphi_j \colon \mathcal{C}ob_f^{(O)}(n+q,-q) \to \mathrm{Imm}^{\varepsilon^1}(n-1,1)$ such that the compositions $\varphi_j \circ \alpha_j$ are the identity maps as follows.

Let j be an integer such that $0 \leq j \leq \lfloor (q-1)/2 \rfloor$. Let $h_j \colon S^{q+1} \to (-\varepsilon, \varepsilon)$ be a Morse function of the $(q+1)$-dimensional sphere into the open interval $(-\varepsilon, \varepsilon)$ with four critical points of index $0, j, j+1, q+1$, respectively, such that the critical value of index j is zero in the interval $(-\varepsilon, \varepsilon)$.

Let j be an integer such that $0 \leq j \leq \lfloor (q-1)/2 \rfloor$.

Definition 3.1. Let $[g \colon M^{n-1} \to \mathbb{R}^n]$ be an element of the group $\mathrm{Imm}^{\varepsilon^1}(n-1,1)$. We identify the framed fibers of the normal bundle of the immersion g with the interval $(-\varepsilon, \varepsilon)$. Let $f_j \colon M^{n-1} \times S^{q+1} \to \mathbb{R}^n$ be a fold map such that for every $p \in M^{n-1}$ the map $f_j |_{\{p\} \times S^{q+1}}$ maps the sphere $\{p\} \times S^{q+1}$ as the Morse function h_j onto the framed fiber $(-\varepsilon, \varepsilon)$ over the point $g(p)$ in $\mathbb{R}^n$ of the normal bundle of the immersion g. Let $\alpha_j([g])$ be the cobordism class of the fold map f_j. In the case of a cobordism between two elements $[g_i]$ $(i = 0, 1)$ of the group $\mathrm{Imm}^{\varepsilon^1}(n-1,1)$ we can do the same construction in order to obtain a cobordism between the fold cobordism classes $\alpha_j[g_i]$ $(i = 0, 1)$. Therefore this clearly defines a homomorphism $\alpha_j \colon \mathrm{Imm}^{\varepsilon^1}(n-1,1) \to \mathcal{C}ob_f^{(O)}(n+q,-q)$.

Definition 3.2. Let $[f \colon Q^{n+q} \to \mathbb{R}^n]$ be an element of the group $\mathcal{C}ob_f^{(O)}(n+q,-q)$. Then let $\varphi_j([f])$ be the cobordism class of the framed immersion $f |_{S_j(f)} \colon S_j(f) \to \mathbb{R}^n$

of the singular set of index j of the fold map f. This clearly defines a homomorphism $\varphi_j\colon \mathcal{C}ob_f^{(O)}(n+q,-q) \to \mathrm{Imm}^{\varepsilon^1}(n-1,1)$.

Lemma 3.3. *The composition*

$$\mathrm{Imm}^{\varepsilon^1}(n-1,1) \xrightarrow{\alpha_j} \mathcal{C}ob_f^{(O)}(n+q,-q) \xrightarrow{\varphi_j} \mathrm{Imm}^{\varepsilon^1}(n-1,1)$$

is the identity map.

Proof. First let us prove the case $j=0$. Let $[g\colon M^{n-1} \to \mathbb{R}^n]$ be an element of the group $\mathrm{Imm}^{\varepsilon^1}(n-1,1)$. Then the framed immersion $\varphi_0(\alpha_0([g]))$ constructed in Definitions 3.1 and 3.2 has three components which are parallel translations of the immersion g, moreover two of these components have the same framing and the third component has an opposite framing. The statement of the lemma follows if we note that the sum of two parallel translations with opposite framing among these three components represents the zero cobordism class in the group $\mathrm{Imm}^{\varepsilon^1}(n-1,1)$. A null-cobordism can be given easily by considering the Stein factorization $\bar{f}_0\colon W_{f_0} \to \mathbb{R}^n$ of the fold map $f_0\colon M^{n-1} \times S^{q+1} \to \mathbb{R}^n$ in Definition 3.1 in the case of $j=0$. The CW complex W_{f_0} is equal to $M^{n-1} \times W_{h_0}$, where W_{h_0} is the source "$\curlywedge$" of the Stein factorization $\bar{h}_0\colon W_{h_0} \to (-\varepsilon, \varepsilon)$ of the Morse function h_0, and the Stein factorization $\bar{f}_0$ maps the subset $M^{n-1} \times \{t\}$ of W_{f_0} $(t \in W_{h_0})$ to $\mathbb{R}^n$ as a parallel translation of the immersion g.

When $j>0$, clearly by the constructions in Definitions 3.1 and 3.2 for a given element $[g]$ of the group $\mathrm{Imm}^{\varepsilon^1}(n-1,1)$ the framed immersion of the singular set of index j of the fold map $f_j\colon M^{n-1} \times S^{q+1} \to \mathbb{R}^n$ equals to g. Hence $\varphi_j(\alpha_j([g])) = [g]$ for every element $[g]$ of the group $\mathrm{Imm}^{\varepsilon^1}(n-1,1)$. □

By the above lemma in order to complete the proof of Theorem 2.3 we have to show that the composition

$$\bigoplus_0^{\lfloor (q-1)/2 \rfloor} \mathrm{Imm}^{\varepsilon^1}(n-1,1) \xrightarrow{\alpha_0+\cdots+\alpha_{\lfloor (q-1)/2 \rfloor}} \mathcal{C}ob_f^{(O)}(n+q,-q) \xrightarrow{\bigoplus_{j=0}^{\lfloor (q-1)/2 \rfloor} \varphi_j} \bigoplus_0^{\lfloor (q-1)/2 \rfloor} \mathrm{Imm}^{\varepsilon^1}(n-1,1)$$

denoted by ρ is an isomorphism. But the homomorphism ρ is clearly injective and surjective since by the definition of the homomorphisms α_j and φ_j $(0 \le j \le \lfloor (q-1)/2 \rfloor)$ the homomorphism ρ has the form

$$\rho(a_0, a_1, \ldots, a_{\lfloor (q-1)/2 \rfloor}) = (a_0, a_1 + a_0, a_2 + a_1, \ldots, a_{\lfloor (q-1)/2 \rfloor} + a_{\lfloor (q-3)/2 \rfloor}).$$

This completes the proof of Theorem 2.3. □

Proof of Theorem 2.4. Let us define two homomorphisms $\beta\colon \mathrm{Imm}(n-1,1) \to \mathcal{C}ob_f(n+q,-q)$ and $\psi\colon \mathcal{C}ob_f(n+q,-q) \to \mathrm{Imm}(n-1,1)$ such that the composition $\psi \circ \beta$ is the identity map as follows.

Let $r\colon \mathbb{R}P^{2k} \to (-\varepsilon,\varepsilon)$ be a Morse function of the $(2k)$-dimensional projective space into the open interval $(-\varepsilon,\varepsilon)$ such that r has $2k+1$ critical points with indeces $0,\ldots,2k$, respectively. We can choose the Morse function r such that there exists a diffeomorphism δ of $\mathbb{R}P^{2k}$ with the properties $r \circ \delta = -r$, $\delta \circ \delta = \mathrm{id}_{\mathbb{R}P^{2k}}$, and that the critical value of index k of the function r is 0 in the interval $(-\varepsilon,\varepsilon)$.

Definition 3.4. Let $[g\colon M^{n-1} \to \mathbb{R}^n]$ be an element of the group $\mathrm{Imm}(n-1,1)$, i.e., the normal bundle $\nu(g)$ of the immersion g is a line bundle induced from the universal line bundle $\gamma^1 \to \mathbb{R}P^\infty$ which we identify with $(-\varepsilon,\varepsilon) \times_{\mathbb{Z}_2} S^\infty \to \mathbb{R}P^\infty$ in the usual way. Let us induce the normal bundle $\nu(g)$ and a manifold $M^{n-1}\bar{\times}\mathbb{R}P^{2k}$ by considering the commutative diagram

$$\begin{array}{ccc}
M^{n-1}\bar{\times}\mathbb{R}P^{2k} & \longrightarrow & \mathbb{R}P^{2k} \times_{\mathbb{Z}_2} S^\infty \\
\Big\downarrow{\scriptstyle \bar{r}} & & \Big\downarrow{\scriptstyle r\times_{\mathbb{Z}_2}\mathrm{id}_{S^\infty}} \\
\nu(g) & \longrightarrow & (-\varepsilon,\varepsilon) \times_{\mathbb{Z}_2} S^\infty \\
\Big\downarrow & & \Big\downarrow \\
M^{n-1} & \longrightarrow & \mathbb{R}P^\infty
\end{array}$$

where the $\mathbb{Z}_2$ action on $\mathbb{R}P^{2k}$ is the action of the diffeomorphism δ (see also [11, Example 3.7]). By the diagram above we obtain a fold map $f' = \mu\circ\bar{r}\colon M^{n-1}\bar{\times}\mathbb{R}P^{2k} \to \mathbb{R}^n$, where μ denotes the immersion of the normal bundle $\nu(g)$ into $\mathbb{R}^n$. Note that for every $p \in M^{n-1}$ the map $f'\mid_{\{p\}\times\mathbb{R}P^{2k}}$ maps the space $\{p\}\times\mathbb{R}P^{2k}$ as the Morse function $\pm r$ onto the fiber $(-\varepsilon,\varepsilon)$ over the point $g(p)$ in $\mathbb{R}^n$ of the normal bundle of the immersion g.

Let $\beta([g])$ be the cobordism class of the fold map $f'\colon M^{n-1}\bar{\times}\mathbb{R}P^{2k} \to \mathbb{R}^n$.

In the case of a cobordism between two elements $[g_i]$ $(i=0,1)$ of the group $\mathrm{Imm}(n-1,1)$ we can do the same construction in order to obtain a cobordism between the elements $\beta[g_i]$ $(i=0,1)$. Therefore this clearly defines a homomorphism $\beta\colon \mathrm{Imm}(n-1,1) \to \mathcal{C}ob_f(n+q,-q)$.

Note that for any $[g] \in \mathrm{Imm}(n-1,1)$ the cobordism classes $\varphi_j(\beta([g]))$ are zero in the group $\mathrm{Imm}^{\varepsilon^1}(n-1,1)$ $(0 \le j \le \lfloor (q-1)/2 \rfloor)$ since for the fold map f' in Definition 3.4 the immersion $f'\mid_{S_j(f')}\colon S_j(f') \to \mathbb{R}^n$ $(0 \le j \le \lfloor (q-1)/2 \rfloor)$ is a double covering of the immersion $f'\mid_{S_k(f')}\colon S_k(f') \to \mathbb{R}^n$ and hence it is a boundary of the immersion of the normal bundle $\nu(f')$ into $\mathbb{R}^n$ (this can be seen easily by considering the Stein factorization of the fold map $f'\colon M^{n-1}\bar{\times}\mathbb{R}P^{2k} \to \mathbb{R}^n$).

Definition 3.5. Let $[f\colon Q^{n+q} \to \mathbb{R}^n]$ be an element of the group $\mathcal{C}ob_f(n+q,-q)$. Let $\psi([f])$ be the cobordism class of the immersion $f\,|_{S_k(f)}\colon S_k(f) \to \mathbb{R}^n$ where $S_k(f)$ denotes the indefinite singular set of index k of the fold map f. This clearly defines a homomorphism $\psi\colon \mathcal{C}ob_f(n+q,-q) \to \mathrm{Imm}(n-1,1)$.

Lemma 3.6. *The composition*

$$\mathrm{Imm}(n-1,1) \xrightarrow{\ \beta\ } \mathcal{C}ob_f(n+q,-q) \xrightarrow{\ \psi\ } \mathrm{Imm}(n-1,1)$$

is the identity map.

Proof. Clearly by the constructions in Definitions 3.4 and 3.5 for a given element $[g]$ of the group $\mathrm{Imm}(n-1,1)$ the immersion of the indefinite singular set of index k of the fold map $f'\colon M^{n-1}\tilde{\times}\mathbb{R}P^{2k} \to \mathbb{R}^n$ equals to g. Hence $\psi(\beta([g])) = [g]$ for every element of the group $\mathrm{Imm}(n-1,1)$. Details are left to the reader. $\square$

The proof of Theorem 2.4 will be finished if we show that the composition

$$\begin{aligned}
&\textstyle\bigoplus_0^{\lfloor (q-1)/2 \rfloor} \mathrm{Imm}^{\varepsilon^1}(n-1,1) \oplus \mathrm{Imm}(n-1,1) \xrightarrow{\alpha_0+\cdots+\alpha_{\lfloor (q-1)/2 \rfloor}+\beta} \\
&\qquad \mathcal{C}ob_f(n+q,-q) \xrightarrow{\bigoplus_{j=0}^{\lfloor (q-1)/2 \rfloor} \varphi_j \oplus \psi} \\
&\textstyle\bigoplus_0^{\lfloor (q-1)/2 \rfloor} \mathrm{Imm}^{\varepsilon^1}(n-1,1) \oplus \mathrm{Imm}(n-1,1)
\end{aligned}$$

denoted by τ is an isomorphism. But the homomorphism τ is clearly injective and surjective since by the definition of the homomorphisms α_j, φ_j $(0 \le j \le \lfloor (q-1)/2 \rfloor)$, β and ψ the homomorphism τ has the form

$$\tau(a_0, a_1, \ldots, a_{\lfloor (q-1)/2 \rfloor}, b) = (a_0, a_1+a_0, a_2+a_1, \ldots, a_{\lfloor (q-1)/2 \rfloor}+a_{\lfloor (q-3)/2 \rfloor}, b+\iota(a_{\lfloor (q-1)/2 \rfloor})),$$

where $\iota\colon \mathrm{Imm}^{\varepsilon^1}(n-1,1) \to \mathrm{Imm}(n-1,1)$ denotes the natural forgetting homomorphism.

This completes the proof of Theorem 2.4. $\square$

4. Bordisms of fold maps

There is also a very geometrical equivalence relation on the set of fold maps $f\colon Q^{n+q} \to N^n$ from closed (oriented) $(n+q)$-dimensional manifolds Q^{n+q} into closed oriented n-dimensional manifolds N^n.

Definition 4.1. (Bordism) Two fold maps $f_i\colon Q_i^{n+q} \to N_i^n$ $(i = 0,1)$ from closed (oriented) $(n+q)$-dimensional manifolds Q_i^{n+1} $(i = 0,1)$ into closed oriented n-dimensional manifolds N_i^n $(i = 0,1)$ are *(oriented) bordant* if

a) there exists a fold map $F\colon X^{n+q+1} \to Y^{n+1}$ from a compact (oriented) $(n+q+1)$-dimensional manifold X^{n+q+1} to a compact oriented $(n+1)$-dimensional manifold Y^{n+1},
b) $\partial X^{n+q+1} = Q_0^{n+q} \amalg (-)Q_1^{n+q}$, $\partial Y^{n+1} = N_0^{n+1} \amalg -N_1^{n+1}$ and
c) $F\,|_{Q_0^{n+q}\times[0,\varepsilon)} = f_0 \times id_{[0,\varepsilon)}$ and $F\,|_{Q_1^{n+q}\times(1-\varepsilon,1]} = f_1 \times id_{(1-\varepsilon,1]}$, where $Q_0^{n+q}\times[0,\varepsilon)$ and $Q_1^{n+q}\times(1-\varepsilon,1]$ are small collar neighbourhoods of ∂X^{n+q+1} with the identifications $Q_0^{n+q} = Q_0^{n+q}\times\{0\}$, $Q_1^{n+q} = Q_1^{n+q}\times\{1\}$.

We call the map F a *bordism* between f_0 and f_1.

We can define a commutative group operation on the set of bordism classes by $[f_0]+[f_1] = f_0 \amalg f_1\colon Q_0^{n+q}\amalg Q_1^{n+q} \to N_0^n \amalg N_1^n$ in the usual way.

By changing the target manifold $\mathbb{R}^n$ to an arbitrary oriented n-dimensional manifold N^n in the previous chapters we obtain analogous theorems about the bordism group of fold maps denoted by $Bor_f^{(O)}(n+q,-q)$.

Let $B\mathrm{Imm}^{\xi^k}(n-k,k)$ denote the usual bordism group of k-codimensional immersions into closed oriented n-manifolds, whose normal bundles can be induced from the vector bundle ξ^k. Note that this group $B\mathrm{Imm}^{\xi^k}(n-k,k)$ is isomorphic to the nth oriented bordism group $\Omega_n(\Gamma_{\xi^k})$ of the classifying space Γ_{ξ^k} for such immersions [15].

Then we have the following theorems which are analogues of Theorems 2.3 and 2.4.

Theorem 4.2. *For $n \geq 1$ and $q > 0$ the bordism group $Bor_f^{(O)}(n+q,-q)$ of fold maps of (oriented) $(n+q)$-dimensional manifolds into closed oriented n-dimensional manifolds contains the direct sum $\bigoplus_0^{\lfloor (q-1)/2 \rfloor} B\mathrm{Imm}^{\varepsilon^1}(n-1,1)$ as a direct summand.*

Theorem 4.3. *For $n \geq 1$, $q > 0$, $k \geq 1$ and $q = 2k-1$ the bordism group $Bor_f(n+q,-q)$ of fold maps of unoriented $(n+q)$-dimensional manifolds into closed oriented n-dimensional manifolds contains the direct sum $B\mathrm{Imm}^{\gamma^1}(n-1,1) \oplus \bigoplus_0^{\lfloor (q-1)/2 \rfloor} B\mathrm{Imm}^{\varepsilon^1}(n-1,1)$ as a direct summand.*

The proofs are analogous to the proofs of Theorems 2.3 and 2.4, respectively.

References

[1] Y. Ando, Cobordisms of maps without prescribed singularities, arXiv:math.GT/0412234v1.
[2] Y. Ando, Stable homotopy groups of spheres and higher singularities, J. Math. Kyoto Univ. **46** (2006) 147–165.
[3] T. Ekholm, A. Szűcs and T. Terpai, Cobordisms of fold maps and maps with prescribed number of cusps, arXiv:math.GT/0701433v1.
[4] K. Ikegami, Cobordism group of Morse functions on manifolds, Hiroshima Math. J. **34** (2004), 211–230.
[5] K. Ikegami and O. Saeki, Cobordism group of Morse functions on surfaces, J. Math. Soc. Japan **55** (2003), 1081–1094.

[6] B. Kalmár, Cobordism group of Morse functions on unoriented surfaces, Kyushu J. Math. **59** (2005), 351–363.

[7] B. Kalmár, Cobordism group of fold maps of oriented 3-manifolds into the plane, to appear.

[8] B. Kalmár, Pontryagin-Thom type construction for negative codimensional singular maps, arXiv:math.GT/0612116v1.

[9] U. Koschorke, Vector fields and other vector bundle morphisms - a singularity approach, Lect. Notes in Math. **847**, Springer-Verlag, 1981.

[10] R. Rimányi and A. Szűcs, Generalized Pontrjagin-Thom construction for maps with singularities, Topology **37** (1998), 1177–1191.

[11] O. Saeki, Notes on the topology of folds, J. Math. Soc. Japan **44** (1992), no. 3, 551–566.

[12] R. Sadykov, Bordism groups of solutions to differential relations, arXiv:math.AT/0608460v1.

[13] R. Sadykov, Cobordism groups of Morin maps, preprint.

[14] A. Szűcs, Analogue of the Thom space for mappings with singularity of type Σ^1 (in Russian), Math. Sb. (N.S.) **108 (150)** (1979), 433–456, 478; English translation: Math. USSR-Sb. **36** (1980), 405–426.

[15] A. Szűcs, Cobordism groups of immersions with restricted self-intersection, Osaka J. Math. **21** (1984), 71–80.

[16] A. Szűcs, Topology of $\Sigma^{1,1}$-singular maps, Math. Proc. Camb. Phil. Soc. **121** (1997), 465–477.

[17] A. Szűcs, On the cobordism group of Morin maps, Acta Math. Hungar. **80** (1998), 191–209.

[18] A. Szűcs, Elimination of singularities by cobordism, Contemporary Mathematics **354** (2004), 301–324.

[19] A. Szűcs, Cobordism of singular maps, arXiv:math.GT/0612152v1.

[20] R. Wells, Cobordism of immersions, Topology **5** (1966), 281–294.

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